How Many Lions Can One Man Avoid?

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Abstract. A pride of lions are prowling among the vertices and edges of an $n \times n$ grid. If their paths are known in advance, is it possible to design a safe path for a man that avoids all lions, assuming that man and lion move at the same speed? In their recent paper [2], Dumitrescu et al. employed probabilistic arguments to show that the number of lions which can always be avoided, $k(n)$, lies in $\Omega(\sqrt{n})$. They raised the question if $k(n) \in \Omega(n)$. Using a proof technique quite different from theirs, we give a positive answer. Even $\lfloor \frac{n}{2} \rfloor$ lions can be avoided in dimension 2. However, there is no escaping from, by order of magnitude, $\Theta(\frac{n^{d-1}}{\sqrt{d}})$ lions on the d-dimensional grid.

Extended Abstract

Pursuit-evasion problems have a long history in mathematics and computer science, and many different models have been studied. At SoCG'07, Dumitrescu et al. [2] introduced a variant that has, apparently, not received much attention before.

Let $G_n^d := (V_n^d, E_n^d)$ denote the d-dimensional $(n \times \ldots \times n)$-grid with vertex set $V_n^d := \{0, \ldots, n-1\}^d$ and edge set $E_n^d := \{\{v, w\} \mid |v - w| = 1\}$ where $|.|$ denotes the L_1-distance, which is defined as the sum of the absolute values of all coordinate differences.

A path π visiting $p \in V_n^d$ at time $t \in \{0, \ldots, T-1\}$ may visit a direct neighbor q of p at time $t+1$, or remain at p. Two paths π_1, π_2 are said to avoid each other if they never occupy the same vertex at the same time t and, in their transition from t to $t+1$, never traverse the same grid edge from opposite sides. Now the problem is the following. What is the maximum number $k = k_d(n)$ such that for all possible sets of k "lion" paths with arbitrary length T in G one can construct a "man" path that avoids them all?

Instead of the path of one man, let us consider the set $W(t)$ of all vertices where this man could be at time t. We may consider $W(t)$ as the set of locations that are, at time t, contaminated by some evil force that spreads one step per time unit in each direction not blocked by a lion. The lions' task is to fight contamination. A lion clears a contaminated vertex by visiting it. Once the lion is gone, the vertex may become recontaminated.

As Dumitrescu et al. [2] observed for $d = 2$, it is easy to verify that

$$k_d(n) + 1 \leq n^{d-1}$$

holds, by sweeping the grid with a hyperplane manned by n^{d-1} lions. Furthermore, Dumitrescu et al. have employed probabilistic arguments in proving that $O(\sqrt{n})$ lions are not able to clear an 2-dimensional $n \times n$ grid. They raised the question if the same result can be shown for a linear number of lions.

We answer this question in the affirmative, by proving the following theorem.

Theorem 1. *$n/2$ lions are not capable of cleaning a 2-dimensional grid of size n.*

Our proof is based on an isoperimetric inequality for grid vertex sets by Bollobás and Leader [1]. We use this inequality in proving the following dynamic saturation property. If the set of cleared vertices has reached a critical size, then its boundary contains, for each lion that clears a vertex in its next move, at least one vertex that is about to be recontaminated. Consequently, the set of cleared vertices cannot increase in size.

As to *d-dimensional grids*, we first demonstrate how 8 lions, rather than $n^{d-1} = 3^2 = 9$, can clear the $3 \times 3 \times 3$ grid. More generally, we prove that the minimum number of lions needed to clear a d-dimensional grid of size n equals, up to a constant factor, the number of grid vertices of L_1-distance $\lfloor \frac{d(n-1)}{2} \rfloor$ from the origin. We call the set of those vertices the *middle slice* because, apart from rounding effects, they have the same L_1-distance to the corner vertices $(0, \ldots, 0)$ and $(n-1, \ldots, n-1)$. To be more precise, if we denote the cardinality of the middle slice in the d-dimensional $(n \times \ldots \times n)$-grid by $m(d, n)$, then the following theorem holds.

Theorem 2. $\lfloor \frac{1}{6} m(d, n) \rfloor - 1 \leq k_d(n) \leq 2m(d, n)$

Since no closed formula for $m(d, n)$ seems to be known, we employ a folk-theorem that establishes an asymptotic estimate by means of the central limit theorem. As a consequence, we obtain that the d-dimensional grid of size n can be cleared by $\Theta(\frac{n^{d-1}}{\sqrt{d}})$ many lions.

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References

1. Béla Bollobás and Imre Leader. Compressions and isoperimetric inequalities. *Journal of Combinatiorial Theory, Series A*, 56:47–62, 1991.
2. A. Dumitrescu, I. Suzuki, and P. Zylinski. Offline variants of the "lion and man" problem. In Jeff Erickson, editor, *Proceedings 23rd Annual Symposium on Computational Geometry*, pages 102–111. ACM, 2007.