

On the Geometric Dilation of Finite Point Sets

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Abstract. Let G be an embedded planar graph whose edges may be curves. For two arbitrary points of G , we can compare the length of the shortest path in G connecting them against their Euclidean distance. The maximum of all these ratios is called the *geometric dilation* of G . Given a finite point set, we would like to know the smallest possible dilation of any graph that contains the given points. In this paper we prove that a dilation of 1.678 is always sufficient, and that $\pi/2 = 1.570\dots$ is sometimes necessary in order to accommodate a finite set of points.

Keywords: Computational geometry, detour, dilation, graph, network, spanner, stretch factor, transportation network.

1 Introduction

Transportation networks like waterways, railroad systems, or urban street systems can be modelled by a graph G in the plane whose edges are piecewise smooth curves that do not intersect, except at vertices of G .¹

The quality of G as means of transport can be measured in the following way. For any two points, p and q , of G , let $\xi_G(p, q)$ denote a shortest path in G from p to q . Then the *dilation* of G is defined by

$$\delta(G) := \max_{p, q \in G} \frac{|\xi_G(p, q)|}{|pq|}. \quad (1)$$

The value of $\delta(G)$ measures the longest possible detour that results from using G instead of moving as the crow flies.

The above definition of $\delta(G)$ does not specify which points p, q of G to consider. There are two alternatives, corresponding to different applications.

Access to a railroad system is only possible at stations. In such a model we would use, as measure of quality, the *graph-theoretic dilation*, where only the *vertices* p, q of G are considered in definition (1). Here, only the lengths of the edges of G are of interest but not their geometric shapes.

Along urban streets, however, houses are densely distributed. Here it makes sense to include *all points* p, q of G in definition (1), vertices and interior edge points alike. This gives rise to the *geometric dilation* of graph G .

¹ That is, we do not allow bridges at this stage, but it would, in principle, be possible to enlarge our model.

The graph-theoretic dilation has been extensively studied in the literature on spanners (see e. g. Eppstein’s chapter in the Handbook of Computational Geometry [6] for a survey). One can efficiently construct spanners of bounded dilation and degree, whose weight is close to that of the minimum spanning tree, see Bose et al. [3]. Also, lower time bounds are known, see Chen et al. [4].

In contrast to this, the geometric dilation is a rather novel concept. So far, there are only three types of results. Icking et al. [9] and Aichholzer et al. [2] have provided upper bounds to the geometric dilation of planar curves in terms of their oscillation width, and Ebberts-Baumann et al. [5], Agarwal et al. [1], and Langerman et al. [11] have shown how to efficiently compute the geometric dilation of a given polygonal chain or cycle over n edges. Recently, Grüne [7] has given an algorithm for the related problem of computing the detour of a simple polygon.

Besides computing the dilation of given graphs, it is quite interesting to construct graphs of low dilation that contain a given finite point set.² In case of the graph-theoretic dilation the optimum solution must be a triangulation, since straight edges work best, and adding edges without creating new vertices never hurts. Yet, it seems not to be known how to efficiently compute the triangulation of minimum graph-theoretic dilation over a given vertex set. It is not even clear what maximum value the lowest possible dilation over all finite point sets can attain (see Problems 8 and 9 in [6]).

In this paper we are addressing the corresponding question for the geometric dilation. Given a finite point set P , we are interested in the smallest possible geometric dilation of any finite planar graph that contains all points of P , i. e., in the value of

$$\Delta(P) := \inf_{P \subset G, G \text{ finite}} \delta(G).$$

We call $\Delta(P)$ the *geometric dilation of the point set P* . Even for a set P of size 3, computing $\Delta(P)$ is a non-trivial task.

Our main interest in this paper is in the maximal value $\Delta(P)$ can attain, for an arbitrary finite point set, P . We are proving the following results.

1. There exist finite point sets whose geometric dilation is as large as $\pi/2 = 1.570\dots$
2. No finite point set can have a dilation larger than 1.678.

The first result is proven in Section 2, using Cauchy’s surface area formula and a novel result on cycles in geometric graphs (Lemma 2). The second result will be shown in Section 3. We shall construct a periodic geometric graph G_∞ of dilation $1.6778\dots$ that covers the plane, such that each finite point set is contained in a finite part G of a scaled copy of G_∞ . While this construction is certainly not efficient—the size of G depends on the rational coordinates of our input set—it serves well in establishing the upper bound.

² Observe that the complete graph over P does not solve this problem because the edge crossings would generate new vertices that must also be considered in definition (1).

2 A lower bound to the geometric dilation of point sets

In this section we show that some point sets can only be embedded in graphs of large geometric dilation. Our main result is the following.

Theorem 1. *Let P_n denote the vertex set of the regular n -gon on the unit circle. Then, we have $\Delta(P_n) = \pi/2 = 1,570\dots$ for each $n \geq 10$.*

In order to prove Theorem 1, we will show that neither any graph with cycles nor a tree containing the given point set has a dilation smaller than $\pi/2$. As preparation we proof the following lemma.

Lemma 1. *Any closed curve C has dilation at least $\pi/2$.*

Proof. First, let C be a closed convex curve, and let δ denote its dilation. For each direction α , there is a unique pair of points (p_α, q_α) , called a *partition pair*, that halves the perimeter $|C|$ of C ; see Figure 1. We call

$$h(\alpha) = |p_\alpha q_\alpha|$$

the *partition distance* at angle α . Let $b(\alpha)$ be the breadth of C in orientation α . Clearly, $b(\alpha) \geq h(\alpha)$ holds. Moreover, we have $\frac{|C|/2}{h(\alpha)} \leq \delta$, by definition of the dilation.

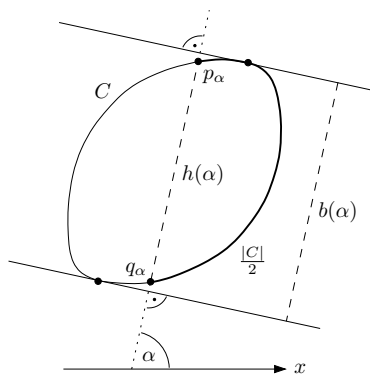


Fig. 1. The breadth of a convex curve is at least its partition distance.

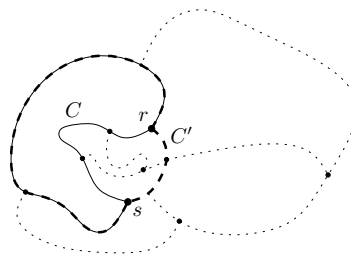


Fig. 2. Cycle C' results from enlarging cycle C .

Thus, by Cauchy's surface area formula,

$$|C| = \int_0^\pi b(\alpha) d\alpha \geq \int_0^\pi h(\alpha) d\alpha \geq \int_0^\pi \frac{|C|/2}{\delta} d\alpha = \frac{\pi|C|}{2\delta},$$

hence $\delta \geq \pi/2$.

Next, let C be a closed non-convex curve. Again, for each orientation α there is a partition pair (p_α, q_α) of C . This can be shown by a continuity argument: Clearly, there is a partition pair (p_β, q_β) for *some* orientation β ; as we let these points move along C in clockwise direction, at equal speed, each will eventually reach its partner's position. During this process, each possible orientation has been attained. Now let $\text{ch}(C)$ denote the convex hull of C . Then, $|C| \geq |\text{ch}(C)|$ holds, and we have $b_{\text{ch}(C)}(\alpha) \geq h_C(\alpha)$. So, the proof for the convex case carries over. \square

Now, let G be an arbitrary geometric graph that contains a bounded face. In order to deal with this case we provide the following result.

Lemma 2. *Let G be a finite geometric graph in the plane that contains a bounded face. Then there exists a cycle, C in G such that for any two points, p and q of C there exists a shortest path $\xi_G(p, q)$ from p to q in G that is a subset of C .*

Proof. (Sketch.) We start with a cycle, C , that equals the boundary of a bounded face. As long as there are shorter connections between points on C than those provided by C , we use them to form a bigger cycle C' (see Figure 2), while maintaining the following.

Invariant. Let a, b be two points on cycle C . Then no shortest path in G connecting them uses any edge that passes through the interior of the bounded face encircled by C . \square

Together with Lemma 1 on the dilation of closed curves the subsequent theorem follows directly:

Theorem 2. *Each graph containing a bounded face has dilation at least $\pi/2$.*

It remains to show, that no graph without cycles, i. e. a tree, can provide a smaller dilation for embedding the vertex set P_n of a regular n -gon with $n \geq 10$.

Lemma 3. *Let tree T contain the point set P_n . Then $\delta(T) \geq \pi/2$ holds.*

Proof. Assume that tree T contains P_n , and that $\delta(T) < \pi/2$ holds. Then, if p, q are two neighboring points of P_n , the unique path $\xi(p, q)$ in T connecting them is of length at most $a\pi/2$, where

$$a = |pq| = 2 \sin\left(\frac{\pi}{n}\right) \leq 2 \sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{2} = 0,618\dots$$

Let z be an arbitrary point on this path. Since z can be reached from p by a curve of length at most $a\pi/2$, its Euclidean distance from p cannot exceed this value. The same holds for q . Thus, z must be included in the lune formed by the two circles of radius

$$a \frac{\pi}{2} \leq \frac{\sqrt{5}-1}{2} \frac{\pi}{2} = 0,9708\dots < 1$$

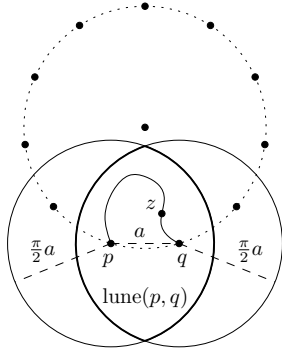


Fig. 3. The path between neighboring points is contained in their lune.

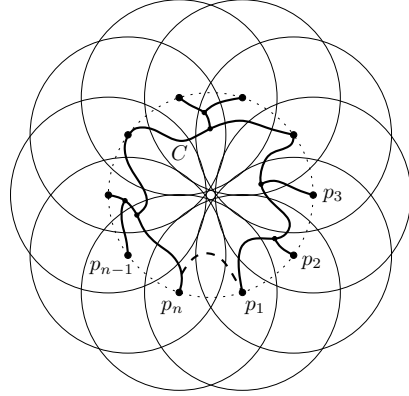


Fig. 4. The path C from p_1 to p_n is not contained in the lune of p_1 and p_n .

centered at p and q ; see Figure 3. Thus, no lune contains the unit circle's center. Now let us consider the arrangement of all lunes of neighboring points, as depicted in Figure 4, and assume that the points are labelled p_1, p_2, \dots, p_n in counterclockwise order. The concatenation

$$C = \xi(p_1, p_2)\xi(p_2, p_3) \dots \xi(p_{n-1}, p_n)$$

is a path in T that is contained in, and visits, all lunes associated with these point pairs. Together with $\xi(p_n, p_1)$, which must be contained in the lune of p_n and p_1 , C forms a closed path in T that encircles the center of the unit circle and is, thus, not contractible, contradicting the fact that T is a tree. \square

Now we can prove Theorem 1.

Proof. Clearly, for each n we have $\Delta(P_n) \leq \pi/2$, because this is the dilation of the unit circle. Let $n \geq 10$. According to Theorem 2, any graph containing a cycle has a dilation $\delta(G) \geq \pi/2$. On the other hand according to Lemma 3 no tree containing P_n can provide a smaller dilation. So we have shown that no graph at all containing all points of P_n can have a dilation smaller than $\pi/2$. \square

The arguments displayed in the proof of Lemma 3 can also be used in proving the following result.

Corollary 1. *Let C be a closed curve, and let P_n be a set of n points evenly placed on C . Then the dilation of any tree containing P_n tends to infinity, as n grows.*

We want to point out that Theorem 1 does not hold for small values of n . Trivially, $\Delta(P_1) = \Delta(P_2) = 1$. For $n = 3$ we have the subsequent result.

Corollary 2. $\Delta(P_3) = \frac{2}{\sqrt{3}} = 1,157\dots$

To show this corollary the following technical lemma, stated here without proof, is useful.

Lemma 4. *Let v be a vertex of G where two edges meet at angle α , that is, the tangents to the piecewise smooth edges in the common vertex v form an angle α . Then, $\delta(G) \geq \frac{1}{\sin(\alpha/2)}$ holds.*

It follows the proof of Corollary 2.

Proof. We can achieve the bound $\Delta(P_3) = \frac{2}{\sqrt{3}}$ by the Steiner tree on P_3 , i. e., by connecting the center of the unit circle by a straight segment to each point of P_3 . These segments meet at an 120° angle. By Lemma 4, this causes a local dilation of $1/\sin(\alpha/2)$. In our case, no bigger value than $1/\sin(60^\circ) = \frac{2}{\sqrt{3}}$ can occur.

In order to prove that no graph G containing P_3 can have a smaller dilation we need only consider the following cases. G cannot contain a cycle, because any cycle has a dilation of at least $\pi/2$ according to Lemma 1. Therefore G has to be either a simple chain or a tree with a vertex of degree at least 3. If G is a simple chain passing through the three points in the order p, q, r , then its dilation is at least $\frac{|pq|+|qr|}{|pr|} = 2$. Otherwise, G is a tree with a vertex of degree at least 3. But then we can be sure that its dilation is at least $\frac{2}{\sqrt{3}}$, by the angle argument of Lemma 4. \square

3 An upper bound to the geometric dilation

In this section we will show that each finite point set can be embedded in a finite graph of geometric dilation at most 1.678. More precisely, we prove the following.

Theorem 3. *There is a periodic, plane covering graph G_∞ of dilation 1,67784... such that each finite set of rational points is contained in a finite part of a scaled copy of G_∞ .*

We proceed in three steps. First, we state a simple yet important technical result in Lemma 5. Then, the proof of Theorem 3 starts with the construction of a certain cycle, C . Graph G_∞ will then be obtained by taking the hexagonal grid of unit length, and replacing each vertex with a copy of C . The proof will be concluded by showing how to embed finite point sets in G_∞ .

In determining the geometric dilation of graphs the following lemma is useful; it has first been used for chains in [5]. Observe that an analogous result for the graph-theoretic dilation does not hold.

Lemma 5. *The geometric dilation of a graph is always attained by two co-visible points.*

Proof. Assume that $\delta(G)$ is attained by points p, q that are not co-visible and have a minimal Euclidean distance, among all such pairs. Then the line segment pq contains a point r of G in its interior. Hence,

$$\begin{aligned} \delta(G) = \frac{\xi_G(p, q)}{|pq|} &\leq \frac{\xi_G(p, r) + \xi_G(r, q)}{|pr| + |rq|} \\ &\leq \max\left(\frac{\xi_G(p, r)}{|pr|}, \frac{\xi_G(r, q)}{|rq|}\right) \\ &\leq \delta(G). \end{aligned}$$

Thus, the dilation of G is also attained by one of the pairs $(p, r), (r, q)$, a contradiction. \square

Now we give the proof of Theorem 3.

Proof. First, we construct a closed cycle, C , that will then be used in building a periodic graph, G_∞ , of dilation $\delta(G) = 1,67784\dots$. The cycle C is defined as follows. We draw the positive X -axis and two more half lines starting from the origin at angles of 120° and -120° , correspondingly. Next, we fix two numbers, $0 < c < b < 0.5$, that will be specified later, when optimizing the bound. Depending on b, c we draw three circles, each of which touches two of the three half lines at the distance $b_1 = \frac{b-c}{2}$ from the origin; see Figure 5. For their

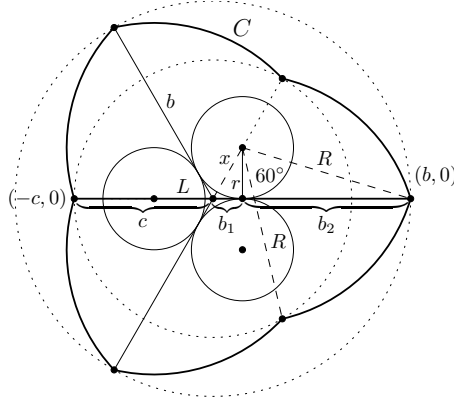


Fig. 5. The cycle C and the essentials of its construction.

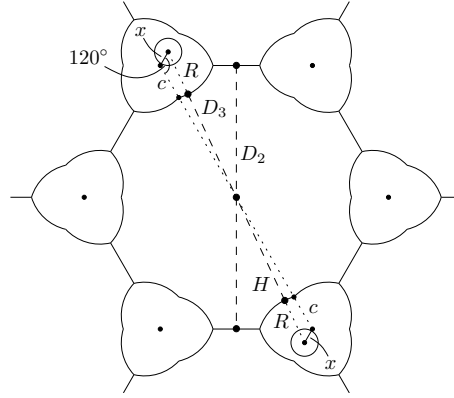


Fig. 6. The periodic graph G_∞ .

radius, r , and for the distance, x from their centers to the origin we obtain

$$\begin{aligned} \frac{x}{2} &= x \cos 60^\circ = b_1 = \frac{b-c}{2} \\ \frac{x\sqrt{3}}{2} &= x \sin 60^\circ = r, \end{aligned}$$

which implies $x = b - c$ and $r = \frac{\sqrt{3}}{2}(b - c)$.

Now we consider the line segment, L , of length $b + c$ from $(-c, 0)$ to $(b, 0)$, and imagine that its midpoint is glued to the upper circle. As this circle rotates clockwise by 60° , the right endpoint of L describes a circular arc of length $R \pi/3$, where

$$\begin{aligned} R &= \sqrt{r^2 + b_2^2} = \sqrt{\frac{3}{4}(b - c)^2 + \frac{1}{4}(b + c)^2} \\ &= \sqrt{b^2 - bc + c^2}. \end{aligned}$$

After this rotation, line segment L is unglued from the upper circle, and glued to the left circle instead, that now rotates clockwise by 60° , and so on. This results in a cycle C that consists of six circular arcs of length $R \pi/3$ each.

By construction, each pair of endpoints of the rotating line segment is a partition pair of C , because the endpoints of C are always moving with identical speed. Moreover, each such pair attains the maximum dilation of C . So, we have

$$D_1 := \delta(C) = \frac{\pi R}{b + c}.$$

Now we construct a periodic graph G_∞ that covers the plane. This graph is obtained by centering rotated copies of cycle C at the vertices of the regular hexagonal grid³ of unit edge length, and cutting off those parts of the axes contained in those copies; see Figure 6.

Thanks to Lemma 5, we need only compute the dilation of the two faces occurring in G_∞ , which are the cycle C and the boundary of the ‘‘dodecagonal’’ face. On the latter, two candidate partition pairs of points exist that might attain maximum dilation. In vertical direction, we have a pair whose dilation equals

$$D_2 = \frac{2\pi R + 3(1 - 2b)}{\sqrt{3}};$$

Observe that the numerator equals three times one third of the perimeter of cycle C , plus three times the length of a shortened unit edge, whereas the denominator measures the height of the hexagonal cell. The other candidate pair is obtained by intersecting, with the copies of C , the line H connecting the centers of two generating circles; see Figure 6. Since the diagonal of the hexagonal cell is of length 2, the distance between these intersection points equals

$$2(\sqrt{x^2 + 1 - 2x \cos 120^\circ} - R),$$

by the law of cosines. This leads to

$$D_3 = \frac{2\pi R + 3(1 - 2b)}{2(\sqrt{x^2 + x + 1} - R)}.$$

³ Without this refinement, the plain hexagonal grid would have a dilation of $\sqrt{3} = 1.7320\dots$

Analysis shows that the maximum of D_1, D_2, D_3 can be minimized to

$$D_1 = D_2 = D_3 = 1.6778\dots$$

by putting

$$c = 0.1248\dots \quad \text{and} \quad b = 0.1939\dots$$

Now let us assume that we are given a finite set of rational points, $P = \{p_1, \dots, p_n\}$, that must be embedded in a graph of low dilation. Assume that

$$p_i = \left(\frac{e_i}{f_i}, \frac{g_i}{h_i} \right)$$

holds with natural numbers e_i, f_i, g_i, h_i . Let

$$K := \left(\frac{1}{2} - b \right) \frac{1}{\sqrt{3}} \frac{1}{\prod_j f_j} \frac{1}{\prod_j h_j} \min_j \frac{f_j}{e_j},$$

where b is the constant involved in constructing the cycle C . By a theorem of Dirichlet's, there are infinitely many integer pairs (r, s) satisfying

$$|\sqrt{3} - \frac{r}{s}| < \frac{1}{s^2} < \frac{K}{s}.$$

We choose one such pair and put

$$\gamma := \frac{1}{r} \frac{1}{\sqrt{3}} \frac{1}{\prod_j f_j} \frac{1}{\prod_j h_j},$$

and construct the graph G introduced above over a hexagonal grid of edge length γ . We are going to place the points p_i on (some of) the horizontal edges of the 12-gons, close to their mid points. This is possible because of two facts. First, each Y -coordinate g_i/h_i is an integer multiple of $\sqrt{3} \gamma$, the distance of two neighboring horizontal levels in G .

Second, each X -coordinate e_i/f_i is close enough to an integer multiple of the distance 3γ between two neighboring mid points of the same Y -level. Indeed,

$$\frac{\sqrt{3}s}{r} \frac{e_i}{f_i} = e_i s \prod_{j \neq i} f_j \prod_j h_j \quad 3\gamma$$

is an integer multiple of 3γ , and we have

$$\begin{aligned} \left| \frac{\sqrt{3}s}{r} \frac{e_i}{f_i} - \frac{e_i}{f_i} \right| &= \frac{e_i}{f_i} \frac{s}{r} |\sqrt{3} - \frac{r}{s}| < \frac{e_i}{f_i} \frac{s}{r} \frac{K}{s} \\ &= \frac{e_i}{f_i} \min_j \frac{f_j}{e_j} \left(\frac{1}{2} - b \right) \gamma < \left(\frac{1}{2} - b \right) \gamma, \end{aligned}$$

so that the point still lies on the horizontal edge. \square

This concludes the proof of Theorem 3 and, thus, of the main result of this section.

4 Concluding remarks

In this paper we have, for the first time, studied the geometric dilation of geometric graphs. We have introduced the notion of the geometric dilation, $\Delta(P)$, of a finite set of points, P , as the minimal dilation of all finite graphs that contain P . We have shown that the vertices of the regular n -gon have dilation $\pi/2 = 1.570\dots$, and that no finite point set has a dilation bigger than 1.678.

These results give rise to many further questions. How can we compute the geometric dilation of a given point set? How costly (in weight and computing time) is the construction of a geometric graph attaining (or: approximating) $\Delta(P)$? What is the precise value of

$$\Delta := \sup_{P \text{ finite}} \Delta(P)?$$

(We conjecture $\Delta > \pi/2$.) And finally, what happens if we extend this definition to non-finite sets, e. g. simple geometric shapes?

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